Interval Observer Approach to Output Stabilization of Time-Varying Input Delay System

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***Abstract*— The output stabilization problem for a linear system with a time-varying input delay is considered. The interval observer technique is extended to delay control systems and applied for obtaining guaranteed interval estimates of the system state. The procedure of the interval observer design, which is based on resolving of the Silvester's equation, is presented. Interval predictor method is introduced in order to design a linear stabilizing feedback. The control design procedure is based on Linear Matrix Inequalities (LMI). The theoretical results are supported by numerical simulations.**

I. Introduction and related works

The time-varying input delay arises in models of control systems due to many reasons. Usually its presence is motivated by a physical nature of a control plant. It may be related with transport delays (like in chemical or hydraulic systems) or computational delay (e.g. in digital controllers or communication networks [16]). On the other hand, time-varying input delay can be introduced "artificially", for example, in order to model a sampling effect [11], [10].

Control of a system with input delay is an important problem treated in a literature (see, for example, [20], [15], [17] and references within). The Smith predictor feedback [22] is a usual tool for control design if the delay is known. This method is well-developed both for constant and for time-varying delay cases, [24], [15]. It has been effectively used even for nonlinear [3] and sliding mode control systems [18]. If delay is constant, but unknown, then estimation technique [4] and/or the delay-adaptive control approach [5] can be applied. This approach is implicitly based on prediction technique. For *unknown input delay* the predictor-based feedback design has to be accompanied with robustness analysis [15], [25].

Typically, the predictor feedback is effectively applicable if the whole state-vector of a system is measured [22], [20], [25], [15], [18]. The observer design for systems with time-varying input and state delay is presented in [21]. The results related to designing of an *output* predictor feedback for systems with input delays, which are known and constant, can be found, for example, in [23] and [14]. The adaptive output feedback regulator for a chain of integrators with an unknown time-varying delay in the input is presented in [6].

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The present paper uses a recently developed technique of interval observers [12], [19] in order to tackle the problem of the output-based control design for linear multiple input and multiple output systems with unknown and time-varying input delay. For the system without delays, the interval observer provides the guaranteed interval estimates of the system state in a real-time. This property helps in controlling the transition processes with respect to system state [7]. This paper extends the interval observer technique to the case of time delay system and presents the interval prediction scheme (interval predictor), that allows us to realize a predictor feedback design based on the LMI technique.

The paper is organized as follows. The next section describes notations to be used in the paper. The section 3 presents the problem statement and basic assumptions. After that, the interval observer and interval predictor are introduced. The control design algorithm is given in the section 5. Finally, a numerical example and conclusions are presented.

II. Notations

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of Hurwitz matrices from $\mathbb{R}^{n\times n}$ is denoted by $\mathcal{H}$.
- The set of Metzler matrices from $\mathbb{R}^{n\times n}$ is denoted by $\mathcal{M}$, i.e.

$$R=\{r_{ij}\}_{i,j=1}^{n}\in\mathcal{M} \quad\Leftrightarrow\quad r_{ij}\geq 0 \text{ for } i\neq j.$$

- The inequality $F\succ 0$ ($F\prec 0$) for a symmetric matrix $F\in\mathbb{R}^{n\times n}$ is meant positive(negative) definiteness of the matrix F. The order relations $F\succeq 0$ and $F\preceq 0$ are used in order to assign the positive and negative semidefiniteness of the matrix F, respectively.
- The inequalities $x>0$, $x<0$, $x\geq 0$ and $x\leq 0$ written for some vector $x\in\mathbb{R}^n$ are to be understood in a componentwise sense.
- The identity matrix of the size $n\times n$ is denoted by I_n; the square zero matrix of the size $n\times n$ is denoted by 0_n; the rectangular zero matrix of the size $n\times m$ is denoted by $0_{n\times m}$.

III. Problem statement

Consider the input delay control system of the form

$$\dot{x}=Ax+Bu(t-h(t)),\quad y=Cx, \tag{1}$$

where $x\in\mathbb{R}^n$ is the system state, $u\in\mathbb{R}^m$ is the vector of control inputs, $y\in\mathbb{R}^k$ is the measured output, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{k\times n}$ are known matrices and input delay $h(t)$ is assumed to be unknown but within the bounded interval:

$$0 \leq \underline{h} \leq h(t) \leq \overline{h}, \tag{2}$$

where the numbers $\underline{h}$ and $\overline{h}$ are given. The system (1) is studied with the initial conditions:

$$\begin{array}{c} x(0) = x_0, \\ u(t) = v(t) \text{ for } t \in [-\overline{h}, 0), \end{array} \tag{3}$$

where $v(t)$ is some bounded function. For simplicity we may assume $v(t) = 0$. The set $\Omega \subset \mathbb{R}^n$ of admissible initial conditions $x_0 \in \Omega$ of the system (1) is assumed to be **bounded** and **known**.

Assumption 1: The pair (A, B) is controllable and the pair (A, C) is observable.

Assumption 2: The information on the control signal $u(t)$ on the time interval $[t - \overline{h}, t)$ can be stored and used for control design purposes.

Remark that the second assumption is usual for a predictor-based approach to control design.

The main objective of this paper is to design a control algorithm for exponential stabilization of the system (1), i.e. for some numbers $c, r > 0$ any solution of the closed-loop system (1) has to satisfy the inequality $\|x(t)\| \leq ce^{-rt}, \forall t > 0$, where $x(0) \in \Omega$.

IV. Interval observer and interval predictor design

A. Interval observer

Let us introduce the following notations

$$\Delta u(\tau, \theta) := u(\tau - \theta) - u(\tau), \tag{4}$$

$$\underline{\Delta} B' u(\tau) := \min_{\theta \in [0, \overline{h} - \underline{h}]} B' \Delta u(\tau, \theta), \tag{5}$$

$$\overline{\Delta} B' u(\tau) := \max_{\theta \in [0, \overline{h} - \underline{h}]} B' \Delta u(\tau, \theta), \tag{6}$$

where $\min(\max)$ is considered componentwise and B' is some matrix of an appropriate dimension.

Lemma 3: Under Assumptions 1-2 there always exist matrices $L \in \mathbb{R}^{n \times k}$ and $S \in \mathbb{R}^{n \times n}, \det(S) \neq 0$ such that

$$A + LC \in \mathcal{H}, \quad S^{-1}(A + LC)S \in \mathcal{M}, \tag{7}$$

and the interval observer of the form

$$\begin{array}{c} \dot{\underline{x}}(t) = \tilde{A}\underline{x} + \tilde{B}u(t - \underline{h}) + \underline{\Delta}\tilde{B}u(t - \underline{h}) + \tilde{L}(\tilde{C}\underline{x}(t) - y(t)), \\ \dot{\overline{x}}(t) = \tilde{A}\overline{x} + \tilde{B}u(t - \underline{h}) + \overline{\Delta}\tilde{B}u(t - \underline{h}) + \tilde{L}(\tilde{C}\overline{x}(t) - y(t)), \\ \underline{x}(0) \leq \tilde{x}(0) \leq \overline{x}(0), \\ \tilde{A} = S^{-1}AS, \tilde{B} = S^{-1}B, \tilde{L} = S^{-1}L, \tilde{C} = CS, \tilde{x} = S^{-1}x, \end{array} \tag{8}$$

guarantees

$$\underline{x}(t) \leq \tilde{x}(t) \leq \overline{x}(t) \quad \forall t > 0, \tag{9}$$

and $\underline{x}(t) \to \tilde{x}(t)$, $\overline{x}(t) \to \tilde{x}(t)$ if $u(t) \to 0$ for $t \to +\infty$.

Proof. I. Since the pair (A, C) is observable then an appropriate selection of the matrix L can assign any real simple negative spectrum of the matrix $A + LC$, i.e. $\sigma(A + LC) = \{\lambda_i\}_{i=1}^n, \lambda_i < 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the matrix S can be defined as Jordan transformation for $A + LC$, which is real in this case. Indeed, $S^{-1}(A + LC)S = \text{diag}(\lambda_i) \in \mathcal{M} \cup \mathcal{H}$. Other variants of computation of S and L are also possible [19].

II. Assumption 2 imply that functions $u(t - \underline{h})$, $\underline{\Delta}\tilde{B}u(t - \underline{h})$ and $\overline{\Delta}\tilde{B}u(t - \underline{h})$ can be calculated for any $t \geq 0$. So, the interval observer (8) is correctly defined. Denote $\underline{e} = \tilde{x} - \underline{x}$ and $\overline{e} = \overline{x} - \tilde{x}$. In this case we have

$$\begin{array}{c} \dot{\underline{e}} = (\tilde{A} + \tilde{L}\tilde{C})\underline{e} + \tilde{B}\Delta u(t - \underline{h}, h(t) - \underline{h}) - \underline{\Delta}\tilde{B}u(t - \underline{h}), \\ \dot{\overline{e}} = (\tilde{A} + \tilde{L}\tilde{C})\overline{e} + \overline{\Delta}\tilde{B}u(t - \underline{h}) - \tilde{B}\Delta u(t - \underline{h}, h(t) - \underline{h}), \end{array} \tag{10}$$

where $\tilde{A} + \tilde{L}\tilde{C} \in \mathcal{M} \cup \mathcal{H}$, $\underline{e}(0) \geq 0$, $\overline{e}(0) \geq 0$. Since

$$\begin{array}{c} \tilde{B}\Delta u(t - \underline{h}, h(t) - \underline{h}) - \underline{\Delta}\tilde{B}u(t - \underline{h}) \geq 0, \\ \overline{\Delta}\tilde{B}u(t - \underline{h}) - \tilde{B}\Delta u(t - \underline{h}, h(t) - \underline{h}) \geq 0, \end{array}$$

then the system (10) is positive [8] and $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t > 0$.

Finally recall that the matrix $\tilde{A} + \tilde{L}\tilde{C}$ is Hurwitz. So, if $u(t) \to 0$ for $t \to +\infty$ then $\underline{e}(t) \to 0$ and $\overline{e}(t) \to 0$. ■

Remark 4: To realize in practice the interval observer (8) the condition $\underline{x}(0) \leq \tilde{x}(0) \leq \overline{x}(0)$ must be guaranteed. Since the set of admissible initial conditions Ω is assumed to be known, the required inequality can be ensured. For example, if $\Omega = \{x \in \mathbb{R}^n : x^T P x < 1\}, P \succ 0$, then $\tilde{x}^T S^T P S \tilde{x} < 1$ and $\underline{x}_i(0) = -\overline{x}_i(0) = -1/\lambda_{\min}(S^T P S)$, $i = 1, 2, ..., n$. Some similar estimates can be also presented if Ω is some polyhedron.

For practical reasons it is important to design interval observer with predefined Metzler matrix. Let some Hurwitz and Metzler matrix R be given and suppose we need to find to find S and L such that

$$S^{-1}(A + LC)S = R.$$

Denote $X = S^{-1}$ and $Y = S^{-1}L$. In this case the required equality can be rewritten in the form of Silvester's equation [2]

$$XA + YC = RX, \tag{11}$$

where $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times k}$.

Proposition 5: [2], [19] If the matrix R has disjoint spectrum and the pair (A, C) is observable then the equation (11) has a solution.

Equation (11) can be rewritten in the form of the system of linear algebraic equations

$$Wz = 0, \tag{12}$$

where

$$\begin{array}{c} W = \begin{pmatrix} I_n \otimes A^T - R \otimes I_n & I \otimes C' \end{pmatrix}, \\ z = (x_{11}, x_{21}, ..., x_{n1}, x_{21}, ..., x_{nn}, y_{11}, ..., y_{n1}, y_{21}, ..., y_{nk})^T, \end{array}$$

where $\otimes$ is the Kroneker product. So, numerically the required solution of the equation (11) can be found as an element of the null space of the matrix W.

B. Interval Predictor

Consider the system (8). By analogy with Artstein transformation [1] let us introduce the following predictor variables:

$$\underline{z}(t) = e^{\tilde{A}\underline{h}}\underline{x}(t) + \int_{-\underline{h}}^{0} e^{-A\theta}\left(\tilde{B}u(t+\theta) + \underline{\Delta}\tilde{B}u(t+\theta)\right)d\theta, \tag{13}$$

$$\overline{z}(t) = e^{\tilde{A}\underline{h}}\overline{x}(t) + \int_{-\underline{h}}^{0} e^{-A\theta}\left(\tilde{B}u(t+\theta) + \overline{\Delta}\tilde{B}u(t+\theta)\right)d\theta, \tag{14}$$

which are correctly defined due to Assumption 2. It is easy to check that in this case the corresponding predictor equations have the form

$$\begin{array}{l} \dot{\underline{z}}(t) = \tilde{A}\underline{z}(t) + \tilde{B}u(t) + \underline{\Delta}\tilde{B}u(t) - e^{\tilde{A}\underline{h}}\tilde{L}\tilde{C}\underline{e}(t), \\ \dot{\overline{z}}(t) = \tilde{A}\overline{z}(t) + \tilde{B}u(t) + \overline{\Delta}\tilde{B}u(t) + e^{\tilde{A}\underline{h}}\tilde{L}\tilde{C}\overline{e}(t). \end{array} \tag{15}$$

Below it is shown that a stabilizing control for the original system can be designed as a linear feedback with respect to predictor variables.

V. Stabilizing control design

Assume that some interval observer (8) for the system (1) is designed and the matrices $S, L, \tilde{A}, \tilde{L}, \tilde{C}, \tilde{B}$ are obtained.

Let us define the control in the form

$$u(t) = Kz(t), \quad z(t) = \frac{\overline{z}(t) + \underline{z}(t)}{2}, \tag{16}$$

where $K \in \mathbb{R}^{m\times n}$.

Remark 6: Let us mention that the control function can be selected in a more general form

$$u(t) = \underline{K}\,\underline{z}(t) + \overline{K}\overline{z}(t),$$

where $\underline{K}, \overline{K} \in \mathbb{R}^{m\times n}$. In particular, the gain matrices can be defined as $\underline{K} = \mu K$ and $\overline{K} = (1-\mu)K$, where $\mu \in [0,1]$. This form of control implies some small changes in formulation and proof of the main theorem given below. We select $\mu = 0.5$ for simplicity and shortness. Moreover, such selection had allowed us to attain the best convergence rate during numerical simulations.

Let $\tilde{B}_i \in \mathbb{R}^{n\times m}, i = 1, 2, ..., n$ be the matrix such that i-th row of $\tilde{B}_i$ coincides with i-th row of the matrix $\tilde{B}$ but all other rows of $\tilde{B}_i$ are zero. Denote also $\tilde{B}_{n+i} = \tilde{B}_i, i = 1, 2..., n$.

Theorem 7: If for some given $\alpha, \beta, \gamma \in \mathbb{R}_+$ the matrices $X, Z, R_i, S_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, ..., 2n$ and the matrix $Y \in \mathbb{R}^{m\times n}$ satisfy the following LMI system

$$\begin{pmatrix} W_e & W_{ez} \\ W_{ez}^T & W_z \end{pmatrix} \preceq 0, X \succ 0, Z \succ 0, R_i \succ 0, S_i \succ 0, \tag{17}$$

where

$$W_e = \begin{pmatrix} \Pi_1 & \tilde{B}_1 Y & ... & \tilde{B}_{2n}Y \\ Y^T\tilde{B}_1^T & -e^{-\beta\Delta h}S_1 & ... & 0 \\ ... & ... & ... & ... \\ Y^T\tilde{B}_{2n}^T & ... & ... & -e^{-\beta\Delta h}S_{2n} \end{pmatrix},$$

$$W_z = \begin{pmatrix} \Pi_2 & \Pi_4 & \tilde{B}_1 Y & ... & \tilde{B}_{2n}Y \\ \Pi_4^T & \Pi_3 & \tilde{B}_1 Y & ... & \tilde{B}_{2n}Y \\ Y^T\tilde{B}_1^T & Y^T\tilde{B}_1^T & -e^{-\alpha\Delta h}R_1 & ... & 0 \\ ... & ... & ... & ... & ... \\ Y^T\tilde{B}_{2n}^T & Y^T\tilde{B}_{2n}^T & 0 & ... & -e^{-\alpha\Delta h}R_{2n} \end{pmatrix}$$

$$W_{ez} = \begin{pmatrix} Z\tilde{C}^T\tilde{L}^T e^{\tilde{A}^T\underline{h}}\left[\begin{array}{cc} I_n & I_n \end{array}\right] & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Pi_1 = (\tilde{A} + \tilde{L}\tilde{C})Z + Z(\tilde{A} + \tilde{L}\tilde{C})^T + \beta Z,$$

$$\Pi_2 = \tilde{A}X + \tilde{B}Y + X\tilde{A}^T + Y^T\tilde{B}^T + \alpha X,$$

$$\Pi_3 = \frac{1}{4}\sum_{i=1}^{2n}(R_i + e^{\gamma\underline{h}}S_i) - \frac{2}{\Delta h}X, \ \Delta h := \overline{h} - \underline{h},$$

$$\Pi_4 = X\tilde{A}^T + Y^T\tilde{B}^T$$

then the system (1) together with the control (16) for

$$K = YX^{-1}$$

is exponentially stable with the convergence rate : $r \geq \min\{\alpha, \beta, \gamma\}$.
Proof of this theorem is given in the Appendix.

It can be shown that under conditions of controllability of the pair $\{A, B\}$ and observability of the pair $\{A, C\}$ the LMI system (17) is feasible for sufficiently small Δh.

VI. Example

A. Linear oscillator

Consider the system (1) with $n = 2, k = m = 1$ and parameters

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

and $\underline{h} = 1$, $\overline{h} = 2$. Solving the Silvester's equation (11) for

$$R = \begin{pmatrix} -44.5200 & 46.0040 \\ 4.4520 & -14.8400 \end{pmatrix}$$

leads to

$$S = 10^3\begin{pmatrix} -0.0592 & 0.0958 \\ -0.4526 & 1.5394 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} 0.9994 \\ -0.0016 \end{pmatrix}.$$

Hence we have

$$\tilde{A} = \begin{pmatrix} 14.6857 & -49.7339 \\ 4.3566 & -14.6857 \end{pmatrix}, \tilde{B} = 10^{-3}\begin{pmatrix} 2.0026 \\ 1.2384 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} -59.2393 & 95.7922 \end{pmatrix}.$$

Finally, using Sedumi-1.3 for MATLAB we solve LMI system (17) for $\alpha = \beta = \gamma = 0.2$ and obtain

$$K = \begin{pmatrix} 137.1771 & -469.0443 \end{pmatrix}.$$

For an interval observer design it is assumed that $x(0) \in \{x \in \mathbb{R}^2 : |x_i| \leq 1, i = 1, 2\}$. Then due to relation $\tilde{x}(0) = S^{-1}x(0)$ the initial conditions for the interval observer (8)

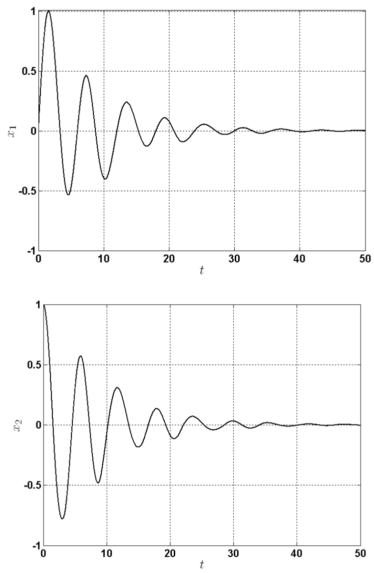

Fig. 1. Evolution of the system states

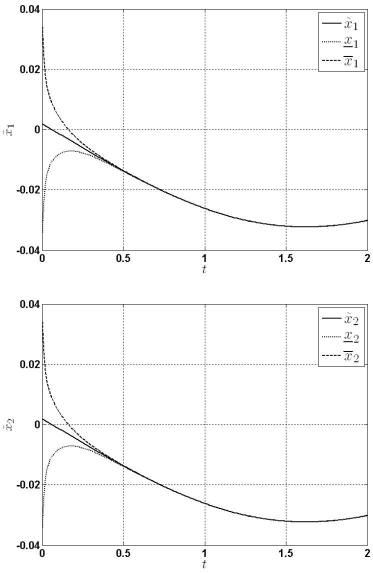

Fig. 2. The real and observed states of the auxiliary vector $\tilde{x}$

can be selected as $\overline{x}(0) = |S^{-1}| \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{x}(0) = -\overline{x}(0)$, where the modulus of the matrix is understood component-wise.

The figure 1 depicts the results of numerical simulations for the system (1) with the control (16) with the delay $h(t) = 1 + 0.5(1 - \text{sign}(\cos(0.5t)))$ the following initial conditions: $x(0) = (0, 1)^T$ and $v(t) = 0$.

The evolution of the observation process for the auxiliary state vector $\tilde{x} = S^{-1}x$ is shown in the figure 2.

B. Double integrator

The adaptive control scheme presented in [6] also admits unknown time varying input delay, but it is applicable only

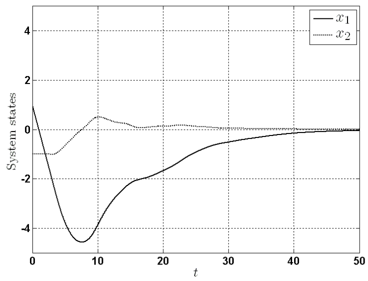

Fig. 3. Evolution of states for controlled double integrator: the case i).

for a chain of integrators. In order to compare our control algorithm with the one presented in [6] we consider the output control problem for double integrator, i.e. $n = 2, k = m = 1$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Solving the Silvester's equation (11) for

$$R = \begin{pmatrix} -3.0000 & 2.3200 \\ 0.2700 & -0.4100 \end{pmatrix}$$

we derive

$$S = \begin{pmatrix} -3.6234 & -0.2599 \\ -1.5558 & -9.1860 \end{pmatrix}, \tilde{L} = \begin{pmatrix} 0.9479 \\ -0.0948 \end{pmatrix}$$

and

$$\tilde{A} = \begin{pmatrix} 0.4346 & 2.5663 \\ -0.0736 & -0.4346 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0.0079 \\ -0.1102 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} -3.6234 & -0.2599 \end{pmatrix}.$$

The numerical simulations have been done for the same time delay as in [6] : i) $h(t) = 2\sin(t) + 2$; ii) $h(t) = 3 + \cos(100t)$; and for the same initial conditions: $x(0) = (1, -1)^T$, $v(t) = 0, t \in [-\overline{h}, 0)$.

In the case i) we have $\underline{h} = 0$, $\overline{h} = 4$. Solving the LMI system (17) for $\alpha = \beta = \gamma = 0.2$ gives

$$K = \begin{pmatrix} 0.7889 & 3.7978 \end{pmatrix}.$$

For the case ii) the estimates of the delay are $\underline{h} = 2$ and $\overline{h} = 4$. The corresponding vector of feedback gains obtained by the LMI system (17) is the following

$$K = \begin{pmatrix} 1.0747 & 4.5060 \end{pmatrix}.$$

The simulations results for control of double integrator are presented in the figures 3 and 4. They show that the control algorithm based on the interval predictor technique provides faster convergence rate of the system states to the origin comparing with the adaptive scheme presented in [6]. Moreover, in contrast to adaptive algorithm it shows a better dumping during the transitory motion.

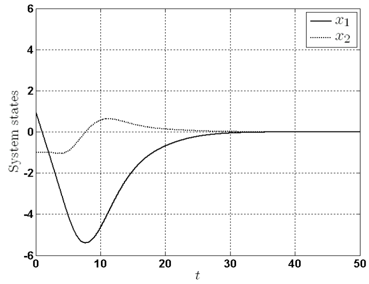

Fig. 4. Evolution of states for controlled double integrator: the case ii).

VII. Conclusions

In the paper an output-based predictor feedback control algorithm is presented for exponential stabilization of linear system with unknown time-varying input delay using interval predictor and interval observer technique. The procedure of the output feedback design requires the solving of the Silvester's equation for observer design and finding solution of the LMI system for adjusting of the feedback control gains. The stability analysis of closed-loop system is based on the method of Lyapunov-Krasovskii functionals.

The main results are presented for linear system with uncertain input delay. However, they can be extended to the case of state and/or output delays and other types of system uncertainties and disturbances. These problems are subjected for future researches.

Appendix

Proof of Theorem 1

I. Due to the form of the control input (16) for $t > \Delta h$ we have

$$\underline{\Delta}\tilde{B}u(t) = \min_{\theta\in[0,\Delta h]} -\tilde{B}K\int_{t-\theta}^{t}\dot{z}(s)ds,$$

$$\overline{\Delta}\tilde{B}u(t) = \max_{\theta\in[0,\Delta h]} -\tilde{B}K\int_{t-\theta}^{t}\dot{z}(s)ds,$$

where $\min(\max)$ is considered in componentwise sense. Then there exist functions $\theta_i : \mathbb{R}_+ \to [0, \Delta h]$, $i = 1, 2, ..., 2n$ such that

$$\left[\underline{\Delta}\tilde{B}u(t)\right]_i = -\tilde{b}_i\mathcal{K}\int_{t-\theta_i(t)}^{t}\dot{z}(s)ds,$$

$$\left[\overline{\Delta}\tilde{B}u(t)\right]_i = -\tilde{b}_i\mathcal{K}\int_{t-\theta_{n+i}(t)}^{t}\dot{z}(s)ds$$

where $\tilde{b}_i$ is the i-th row of the matrix $\tilde{B}$.

Introduce the auxiliary state vector

$$e = \frac{\overline{e}-\underline{e}}{2}. \tag{18}$$

In this case using (10), (15) for $t \geq \overline{h}$ we obtain the following system

$$\begin{aligned}
\dot{e}&=(\tilde{A}+\tilde{L}\tilde{C})e(t) - \tfrac{1}{2}\sum_{i=1}^{2n}\tilde{B}_iK\int_{t-\underline{h}-\theta_i(t-\underline{h})}^{t-h(t)}\dot{z}(s)ds,\\
\dot{z}&=(\tilde{A}+\tilde{B}K)z(t) - \tfrac{1}{2}\sum_{i=1}^{2n}\tilde{B}_iK\int_{t-\theta_i(t)}^{t}\dot{z}(s)ds+e^{\tilde{A}\underline{h}}\tilde{L}\tilde{C}e(t).
\end{aligned} \tag{19}$$

II. Consider the Lyapunov-Krasovski functional(LKF) defined for $t \geq \overline{h}$

$$\begin{gathered}
V(t,e(t),z(t),\dot{z}(\cdot)) = V_e+V_z+V_{ez},\\
V_e(t,e(t),\dot{z}(\cdot))=e^T(t)Qe(t)+\\
\Delta h\sum_{i=1}^{2n}\int_{-\Delta h}^{0}\int_{t-\underline{h}+\theta}^{t-\underline{h}} e^{\beta(s-t+\underline{h})}\dot{z}^T(s)\tilde{S}_i\dot{z}(s)ds\,d\theta,\\
V_z(t,z(t),\dot{z}(\cdot)) = z^T(t)Pz(t)+\\
\Delta h\sum_{i=1}^{2n}\int_{-\Delta h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds\,d\theta,\\
V_{ez}(\dot{z}(\cdot))=(\Delta h)^2\sum_{i=1}^{2n}\int_{t-\underline{h}}^{t} e^{\gamma(s-t+\underline{h})}\dot{z}^T(s)\tilde{S}_i\dot{z}(s)ds
\end{gathered} \tag{20}$$

where $\alpha,\beta,\gamma \in \mathbb{R}_+$, $P, Q, \tilde{S}_i, \tilde{R}_i \in \mathbb{R}^{n\times n}, P \succ 0, Q \succ 0, \tilde{S}_i \succ 0, \tilde{R}_i \succ 0, i = 1, 2, ..., 2n$.

The structure of term V_z of the LKF V is similar to the one from the paper [10]. The terms V_{ez} and V_e are motivated by the extended system (19), that implicitly contains both system state and observer state.

Remark also that the presented functional V has the form of a discretized LKF [13]. In contrast to usual discretization scheme applied to a complete LKF [13], which was introduced artificially in order to obtain stability conditions in LMI forms, the discretized LKF of this paper is imposed by structures of the interval observer and the interval predictor.

III. Calculate the time derivative of the functional V_z:

$$\begin{aligned}
\dot{V}_z &= 2z^T(t)P\dot{z}(t)\\
&- \alpha\Delta h\sum_{i=1}^{2n}\int_{-\Delta h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds\,d\theta\\
&\quad +(\Delta h)^2\sum_{i=1}^{2n}\dot{z}^T(t)\tilde{R}_i\dot{z}(t)\\
&- \Delta h\sum_{i=1}^{2n}\int_{t-\Delta h}^{t} e^{\alpha(s-t)}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds\\
&= -\alpha V_z(t,z(t),\dot{z}(\cdot)) + \alpha z^T(t)Pz(t) + 2z^T(t)P\dot{z}(t)\\
&+ (\Delta h)^2\sum_{i=1}^{2n}\dot{z}^T(t)\tilde{R}_i\dot{z}(t)\\
&- \Delta h\sum_{i=1}^{2n}\int_{t-\Delta h}^{t} e^{\alpha(s-t)}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds.
\end{aligned}$$

On the one hand we have

$$\tfrac{1}{e^{\alpha\Delta h}}\int_{t\text{-}\Delta h}^{t}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds \leq \int_{t\text{-}\Delta h}^{t} e^{\alpha(s\text{-}t)}\dot{z}^T(s)\tilde{R}_i\dot{z}(s)ds.$$

On the other hand due to $0 \le \theta_i(t) \le \Delta h$ the following inequalities

$$\int_{t-\theta_i(t)}^{t} \dot z^T(s)\tilde R_i \dot z(s)ds \le \int_{t-\Delta h}^{t} \dot z^T(s)\tilde R_i \dot z(s)ds,$$

$$\theta_i(t)\int_{t-\theta_i(t)}^{t} \dot z^T(s)\tilde R_i \dot z(s)ds \le \Delta h\int_{t-\theta_i(t)}^{t} \dot z^T(s)\tilde R_i \dot z(s)ds$$

hold. So, taking into account Jensen's Inequality

$$\left(\int_{t-\theta_i(t)}^{t} \dot z(s)ds\right)^T \tilde R_i \left(\int_{t-\theta_i(t)}^{t} \dot z(s)ds\right) \le \theta_i(t)\int_{t-\theta_i(t)}^{t} \dot z^T(s)\tilde R_i \dot z(s)ds$$

we derive

$$\begin{array}{rcl}\dot V_z & \le & -\alpha V_z + \alpha z^T(t)Pz(t)\\ & + & (\Delta h)^2 \dot z^T(t)\left(\sum\limits_{i=1}^{2n}\tilde R_i\right)\dot z(t) + 2z^T(t)P\dot z(t)\\ & - & \frac{1}{e^{\alpha\Delta h}}\sum\limits_{i=1}^{2n}\left(\int\limits_{t-\theta_i(t)}^{t}\dot z(s)ds\right)^T \tilde R_i \left(\int\limits_{t-\theta_i(t)}^{t}\dot z(s)ds\right)\\ & = & -\alpha V_z(t,z(t),\dot z(\cdot)) + g_z^T W_z^1 g_z\\ & & -(\Delta h)^2\dot z^T(t)\left(\sum\limits_{i=1}^{2n} e^{\gamma \underline{h}}\tilde S_i\right)\dot z(t),\end{array}$$

where

$$g_z = \begin{pmatrix} z(t)\\ \dot z(t)\\ \int\limits_{t-\theta_1(t)}^{t}\dot z(s)ds\\ \ldots\\ \int\limits_{t-\theta_{2n}(t)}^{t}\dot z(s)ds\end{pmatrix},$$

$$W_z^1 = \begin{pmatrix} \alpha P & P & 0_n & \ldots & 0_n\\ P & M & 0_n & \ldots & 0_n\\ 0_n & 0_n & -e^{-\alpha\Delta h}\tilde R_1 & \ldots & 0_n\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0_n & 0_n & 0_n & \ldots & -e^{-\alpha\Delta h}\tilde R_{2n}\end{pmatrix},$$

where $M := (\Delta h)^2\sum\limits_{i=1}^{2n}(\tilde R_i + e^{\gamma\underline{h}}\tilde S_i)$. Similar considerations for the functional $V_e(t,e(t),\dot z(\cdot))$ give

$$\begin{array}{rcl}\dot V_e & \le & -\beta V_e(t,z(t),\dot z(\cdot)) + g_e^T W_e^1 g_e + 2e^T Q\dot e(t)\\ & & +(\Delta h)^2 \dot z^T(t-\underline{h})\left(\sum\limits_{i=1}^{2n}\tilde S_i\right)\dot z(t-\underline{h}),\end{array}$$

where

$$g_e = \begin{pmatrix} e(t)\\ \int\limits_{t-\underline{h}-\theta_1(t-\underline{h})}^{t-\underline{h}}\dot z(s)ds\\ \ldots\\ \int\limits_{t-\underline{h}-\theta_{2n}(t-\underline{h})}^{t-\underline{h}}\dot z(s)ds\end{pmatrix},$$

$$W_e^1 = \begin{pmatrix} \beta Q & 0_n & \ldots & 0_n\\ 0_n & -e^{-\beta\Delta h}\tilde S_1 & \ldots & 0_n\\ \ldots & \ldots & \ldots & \ldots\\ 0_n & \ldots & \ldots & -e^{-\beta\Delta h}\tilde S_{2n}\end{pmatrix}.$$

Finally, taking into account

$$\begin{array}{c}\frac{d}{dt}\int\limits_{t-\underline{h}}^{t} e^{\gamma(s-t+\underline{h})}\dot z^T(s)\tilde S_i\dot z(s)ds =\\ -\gamma\int\limits_{t-\underline{h}}^{t} e^{\gamma(s-t+\underline{h})}\dot z^T(s)\tilde S_i\dot z(s)ds\\ +e^{\gamma\underline{h}}\dot z^T(t)\tilde S_i\dot z(t) - \dot z^T(t-\underline{h})\tilde S_i\dot z(t-\underline{h})\end{array}$$

we conclude

$$\begin{array}{c}\dot V(t,e(t),z(t),\dot z(\cdot)) \le -\min\{\alpha,\beta,\gamma\}V(t,e(t),z(t),\dot z(\cdot))\\ +\begin{pmatrix} g_e\\ g_z\end{pmatrix}^T \Phi \begin{pmatrix} g_e\\ g_z\end{pmatrix} + 2e^T Q\dot e(t),\end{array}$$

where

$$\Phi := \begin{pmatrix} W_e^1 & 0_{(n+2n^2)\times(2n+2n^2)}\\ 0_{(2n+2n^2)\times(n+2n^2)} & W_z^1\end{pmatrix}.$$

IV. Following the descriptor approach [9] we consider the following equality

$$\begin{array}{c}0 = 2e^T(t)Q\times\\ \left((\tilde A + \tilde L\tilde C)e(t) - \frac{1}{2}\sum\limits_{i=1}^{2n}\tilde B_i K\int\limits_{t-\underline{h}-\theta_i(t-\underline{h})}^{t-h(t)}\dot z(s)ds - \dot e(t)\right)\\ +2(Pz(t)+\Delta hP\dot z(t))^T\times\\ \left((\tilde A+\tilde BK)z(t) - \sum\limits_{i=1}^{2n}\tilde B_i K\int\limits_{t-\theta_i(t)}^{t}\frac{\dot z(s)}{2}ds + e^{\tilde A\underline{h}}\tilde L\tilde C e(t) - \dot z(t)\right)\end{array}$$

that obviously holds for any solution $(e(t), z(t))$ of the system (19) if $t > \overline{h}$. This equality can be rewritten in the form

$$0 = \begin{pmatrix} g_e\\ g_z\end{pmatrix}^T \begin{pmatrix} W_e^2 & W_0\\ W_0^T & W_z^2\end{pmatrix}\begin{pmatrix} g_e\\ g_z\end{pmatrix} - 2e^T(t)Q\dot e(t),$$

where

$$W_e^2 = \begin{pmatrix} Q(\tilde A+\tilde L\tilde C) + (\tilde A+\tilde L\tilde C)^T Q & \frac{-Q\tilde B_1 K}{2} & \ldots & \frac{-Q\tilde B_{2n}K}{2}\\ \frac{-K^T\tilde B_1^T Q}{2} & 0_n & \ldots & 0_n\\ \ldots & \ldots & \ldots & \ldots\\ \frac{-K^T\tilde B_{2n}^T Q}{2} & 0_n & \ldots & 0_n\end{pmatrix},$$

$$W_z^2 = \begin{pmatrix} \Psi_1 & \Psi_2 & \frac{-P\tilde B_1 K}{2} & \ldots & \frac{-P\tilde B_{2n}K}{2}\\ * & -2\Delta hP & \frac{-\Delta hP\tilde B_1 K}{2} & \ldots & \frac{-\Delta hP\tilde B_{2n}K}{2}\\ * & * & 0_n & \ldots & 0_n\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ * & * & * & \ldots & 0_n\end{pmatrix},$$

where $\Psi_1 = P(\tilde A + \tilde BK) + (\tilde A + \tilde BK)^T P$ and $\Psi_2 = \Delta h(\tilde A + \tilde BK)^T P - P$ and $*$ replaces the corresponding symmetric block. Hence, the time derivative of the functional

V calculated along the trajectories of the system (19) can be estimated as

$$\dot{V}(t,e(t),z(t),\dot{z}(\cdot)) \leq -rV(t,e(t),z(t),\dot{z}(\cdot)) + \begin{pmatrix} g_e \\ g_z \end{pmatrix}^T \begin{pmatrix} W_e^1+W_e^2 & W_0 \\ W_0^T & W_z^1+W_z^2 \end{pmatrix} \begin{pmatrix} g_e \\ g_z \end{pmatrix},$$

where $r = \min\{\alpha,\beta,\gamma\}$ and

$$W_0 = \begin{pmatrix} \tilde{C}^T \tilde{L}^T e^{\tilde{A}^T \underline{h}} \begin{bmatrix} P & \Delta h P \end{bmatrix} & 0_{n\times(2n^2)} \\ 0_{(2n^2)\times 2n} & 0_{2n^2} \end{pmatrix}.$$

Finally, for

$$Z{=}Q^{-1}, X{=}P^{-1}, \tilde{S}_i{=}\frac{1}{4}PS_iP, \tilde{R}_i{=}\frac{1}{4}PR_iP, Y{=}KP^{-1}$$

we have

$$G^T \begin{pmatrix} W_e^1+W_e^2 & W_0 \\ W_0^T & W_z^1+W_z^2 \end{pmatrix} G = \begin{pmatrix} W_e & W_{ez} \\ W_{ez}^T & W_z \end{pmatrix},$$

where

$$G = \begin{pmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & -2I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\Delta h} I_{2n} & 0 \\ 0 & 0 & 0 & 0 & -2I_{4n^2} \end{pmatrix} \times \begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & P^{-1} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & P^{-1} \end{pmatrix}.$$

Therefore, feasibility of LMI (17) implies exponential stability of (19).

V. On the one hand, by definition we have

$$e(t) = \frac{\overline{e}(t) - \underline{e}(t)}{2} = \frac{\overline{x}(t) + \underline{x}(t)}{2} - \tilde{x}(t).$$

On the other hand,

$$z(t) = \frac{\overline{z}(t) + \underline{z}(t)}{2} = e^{\tilde{A}\underline{h}} \frac{\underline{x}(t) + \overline{x}(t)}{2} + \int_{-\underline{h}}^{0} e^{-\tilde{A}\theta} \left(\tilde{B}u(t+\theta) + \frac{\underline{\Delta}\tilde{B}u(t+\theta) + \overline{\Delta}\tilde{B}u(t+\theta)}{2} \right) d\theta,$$

where $u(t) = Kz(t)$. Hence, the limits $e(t) \to 0$ and $z(t) \to 0$ imply $\tilde{x}(t) \to 0$ or, equivalently, $x(t) \to 0$. Moreover, it can be easily shown that the rate of convergence of $x(t)$ to zero is the same as for $e(t)$ and $z(t)$.

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